

Polygonal Derivation of the Neutrino Mass Matrix

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Abstract

Representations of the symmetry group D_n of the n -sided regular polygon have generic multiplication rules if n is prime. Using D_n with $n = 5$ or greater, a particular well-known form of the Majorana neutrino mass matrix is derived.

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The form of the 3×3 Majorana neutrino mass matrix \mathcal{M}_ν has been the topic of theoretical study for some time. If \mathcal{M}_ν has less than the full 6 parameters, then there exists at least one relationship among masses and mixing angles, which may be tested against the increasingly more precise experimental data from neutrino oscillations. However, even if such a comparison is successful, the question still remains as to why it has such a form. A possible answer is that it comes from an underlying symmetry. In this paper, it is shown how

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix} \quad (1)$$

may be derived from D_n , the symmetry group of the n -sided regular polygon, where n is a prime number, equal to or greater than 5.

Consider D_5 , the symmetry group of the regular pentagon. It has 10 elements, 4 equivalence classes, and 4 irreducible representations. Its character table is given by

Table 1: Character Table of D_5 .

class	n	h	χ_1	χ_2	χ_3	χ_4
C_1	1	1	1	1	2	2
C_2	5	2	1	-1	0	0
C_3	2	5	1	1	$\phi - 1$	$-\phi$
C_4	2	5	1	1	$-\phi$	$\phi - 1$

Here n is the number of elements and h is the order of each element. The number ϕ is the Golden Ratio (or Divine Proportion) known to the ancient Greeks:

$$\phi = \frac{\sqrt{5} + 1}{2} \simeq 1.618, \quad (2)$$

and satisfies the equation

$$\phi^2 = \phi + 1, \quad (3)$$

which implies that

$$\phi^{k+1} = \phi F_{k+1} + F_k, \quad (4)$$

where F_k are the Fibonacci numbers. [Zadar on the Dalmatian coast in Croatia is an ancient city with a rich history and a university whose origin dates back to 1396. One person who taught there was Luca Pacioli, whose famous work *Divina Proportione* (1509) was illustrated by Leonardo da Vinci.]

The character of each representation is its trace and must satisfy the following two orthogonality conditions:

$$\sum_{C_i} n_i \chi_{ai} \chi_{bi}^* = n \delta_{ab}, \quad \sum_{\chi_a} n_i \chi_{ai} \chi_{aj}^* = n \delta_{ij}, \quad (5)$$

where n is the total number of elements. The number of irreducible representations must be equal to the number of equivalence classes.

The two irreducible two-dimensional representations of D_5 may be chosen as follows. For **2**, let

$$\begin{aligned} C_1 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_2 : \begin{pmatrix} 0 & \omega^k \\ \omega^{5-k} & 0 \end{pmatrix}, \quad (k = 0, 1, 2, 3, 4); \\ C_3 : \begin{pmatrix} \omega & 0 \\ 0 & \omega^4 \end{pmatrix}, \quad \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega \end{pmatrix}, \quad C_4 : \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix}, \quad \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^2 \end{pmatrix}, \end{aligned} \quad (6)$$

where $\omega = \exp(2\pi i/5)$, then **2'** is simply obtained by interchanging C_3 and C_4 . Note that

$$2 \cos(2\pi/5) = \phi - 1, \quad 2 \cos(4\pi/5) = -\phi, \quad (7)$$

as expected.

For D_n with n prime, there are $2n$ elements divided into $(n+3)/2$ equivalence classes: C_1 contains just the identity, C_2 has the n reflections, C_k from $k = 3$ to $(n+3)/2$ has 2 elements each of order n . There are 2 one-dimensional representations and $(n-1)/2$ two-dimensional ones. For $D_3 = S_3$, the above reduces to the “complex” representation with $\omega = \exp(2\pi i/3)$ discussed in a recent review [1].

The group multiplication rules of D_5 are:

$$\mathbf{1}' \times \mathbf{1}' = \mathbf{1}, \quad \mathbf{1}' \times \mathbf{2} = \mathbf{2}, \quad \mathbf{1}' \times \mathbf{2}' = \mathbf{2}', \quad (8)$$

$$\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}', \quad \mathbf{2}' \times \mathbf{2}' = \mathbf{1} + \mathbf{1}' + \mathbf{2}, \quad \mathbf{2} \times \mathbf{2}' = \mathbf{2} + \mathbf{2}'. \quad (9)$$

In particular, let $(a_1, a_2), (b_1, b_2) \sim \mathbf{2}$, then

$$a_1 b_2 + a_2 b_1 \sim \mathbf{1}, \quad a_1 b_2 - a_2 b_1 \sim \mathbf{1}', \quad (a_1 b_1, a_2 b_2) \sim \mathbf{2}'. \quad (10)$$

Similarly, in the decomposition of $\mathbf{2}' \times \mathbf{2}'$, $(a'_2 b'_2, a'_1 b'_1) \sim \mathbf{2}$, and in the decomposition of $\mathbf{2} \times \mathbf{2}'$, $(a_2 a'_1, a_1 a'_2) \sim \mathbf{2}$, and $(a_2 a'_2, a_1 a'_1) \sim \mathbf{2}'$.

The most natural assignment of the 3 lepton families under D_5 is

$$(\nu_i, l_i), \quad l_i^c \sim \mathbf{1} + \mathbf{2}. \quad (11)$$

Assuming two Higgs doublets $\Phi_1 \sim \mathbf{1}$, $\Phi_2 \sim \mathbf{1}'$, the charged-lepton mass matrix is then of the form

$$\mathcal{M}_l = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b - c \\ 0 & b + c & 0 \end{pmatrix}, \quad (12)$$

where a, b come from $\langle \phi_1^0 \rangle$, and c from $\langle \phi_1^0 \rangle$. Redefining $l_{2,3}^c$ as $l_{3,2}^c$, \mathcal{M}_l becomes diagonal with $m_e = |a|$, $m_\mu = |b - c|$, $m_\tau = |b + c|$.

Assuming that neutrino masses are Majorana and that they come from the naturally small vacuum expectation values [2] of heavy Higgs triplets $\xi_1 \sim \mathbf{1}$, $\xi_{2,3} \sim \mathbf{2}$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix} \quad (13)$$

as advertised, where a, b come from $\langle \xi_1^0 \rangle$, and $c = f \langle \xi_3^0 \rangle$, $d = f \langle \xi_2^0 \rangle$. The two texture zeros are the result of the absence of a Higgs triplet transforming as $\mathbf{2}'$. In the case of $D_3 = S_3$, there is only one two-dimensional representation, hence these zeros cannot be maintained without also making $c = d = 0$.

The decomposition $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}'$ holds not only in D_5 , but also in D_n with n prime and $n > 5$. For example in D_7 , there are 3 two-dimensional irreducible representations, corresponding to the 3 cyclic permutations of

$$C_3 : \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix}, \quad (14)$$

$$C_4 : \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^5 \end{pmatrix}, \quad \begin{pmatrix} \omega^5 & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad (15)$$

$$C_5 : \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^4 \end{pmatrix}, \quad \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^3 \end{pmatrix}, \quad (16)$$

where $\omega = \exp(2\pi i/7)$. It is clear that

$$\mathbf{2}_1 \times \mathbf{2}_1 = \mathbf{1} + \mathbf{1}' + \mathbf{2}_2, \quad \mathbf{2}_2 \times \mathbf{2}_2 = \mathbf{1} + \mathbf{1}' + \mathbf{2}_3, \quad (17)$$

etc. Hence Eq. (13) is valid in all these symmetries.

Phenomenologically, Eq. (13) has been studied [3] as an example of the class of neutrino mass matrices with two texture zeros. It was first derived from a symmetry (Q_8 or D_4) only recently [4]. Whereas Q_8 or D_4 allows other forms, D_n with n prime and $n \geq 5$ allows only Eq. (13). Models based on $D_4 \times Z_2$ have also been proposed [5]. The 4 parameters of Eq. (13) imply that $m_{1,2,3}$ are related to the mixing angles. Given the present global experimental constraints [6]:

$$\Delta m_{atm}^2 = (1.5 - 3.4) \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{atm} > 0.92, \quad (18)$$

$$\Delta m_{sol}^2 = (7.7 - 8.8) \times 10^{-5} \text{ eV}^2, \quad \tan^2 \theta_{sol} = 0.33 - 0.49, \quad (19)$$

and $|\sin \theta_{13}| < 0.2$, the allowed region in the $m_3 - m_2$ plane has been obtained in Ref. [4]. That figure is reproduced here for the convenience of the reader. It shows that there are lower bounds on m_2 and m_3 and that $m_3 < m_2$ up to about 0.1 eV. The parameter a in Eq. (13) measures neutrinoless double beta decay and has a lower bound of about 0.02 eV in this case.

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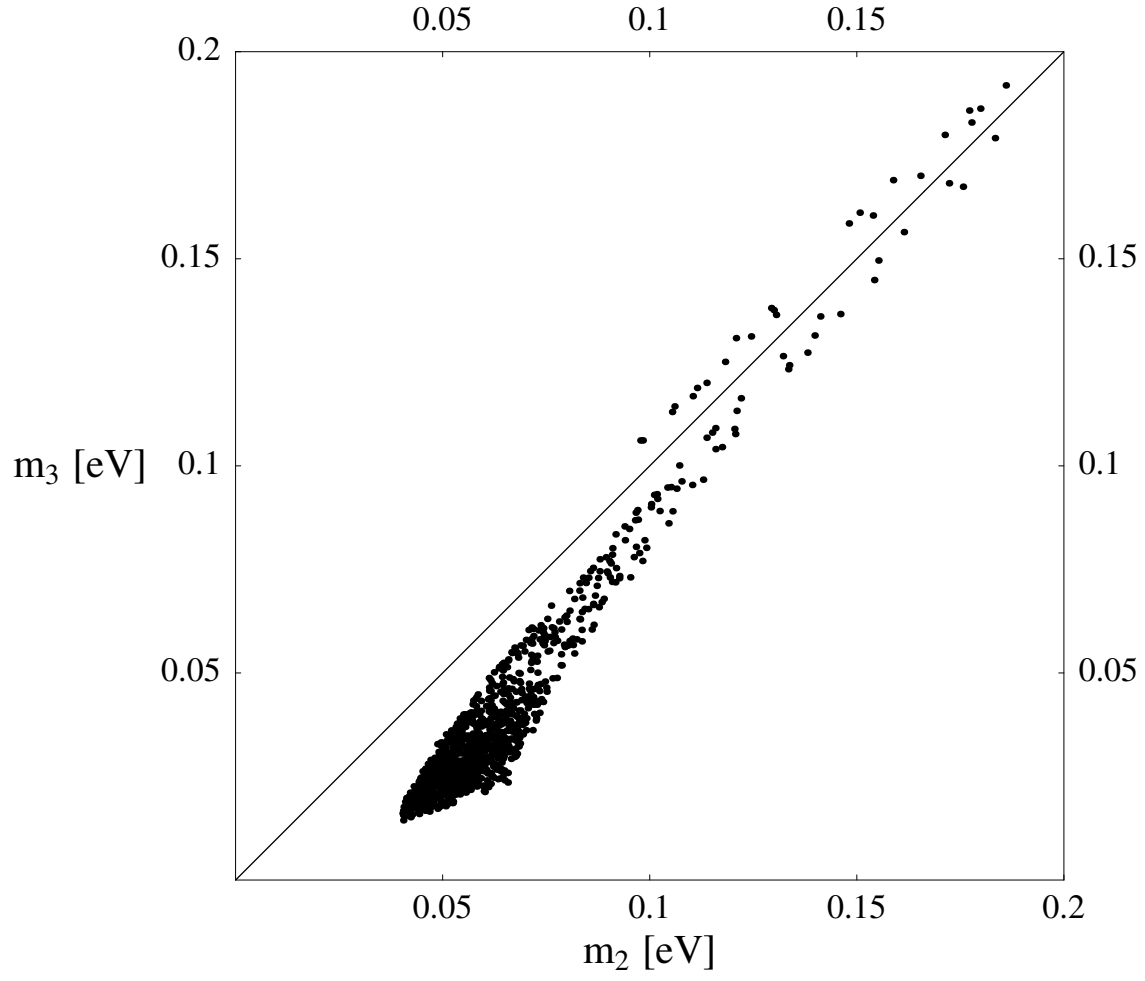


Figure 1: Allowed region in $m_2 - m_3$ plane for Eq. (13)